Solutions 8

Exercise 2.4

a) The likelihood ratio function takes the form

$$L(y) = \frac{f_Y(y|H_1)}{f_Y(y|H_0)} = \sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}y^2 + |y|} \tag{1}$$

b) Firstly, plot the curve as follows:

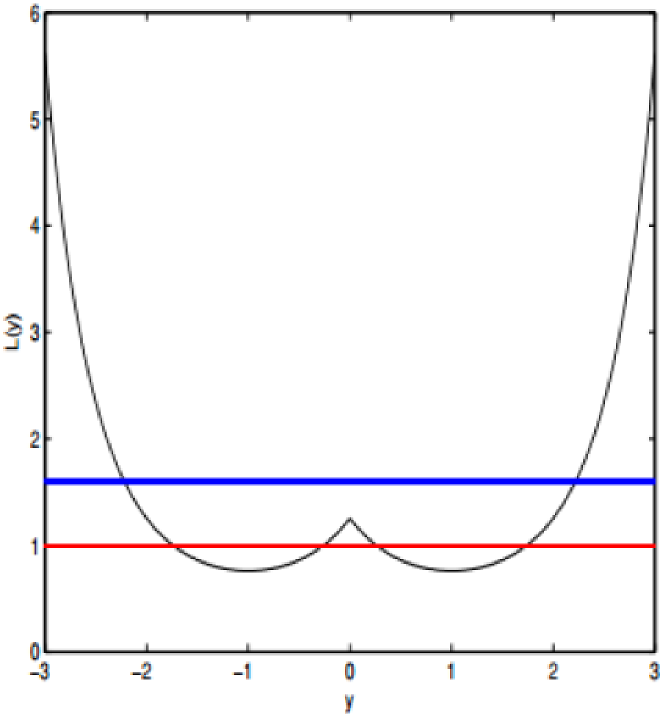


Figure 1: Plot of likelihood ratio function $L(y)$.

- 1) When $\tau \le \sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}}$, the decision is H_1 whatever the y is.
- 2) When $\sqrt{\frac{\pi}{2}} e^{-\frac{1}{2}} \le \tau \le \sqrt{\frac{\pi}{2}}$, the decision is

$$\begin{cases} \text{decision} = 1 & y \in (-\infty, -1-\alpha) \bigcup (\alpha - 1, 1 - \alpha) \bigcup (1+\alpha, +\infty) \\ \text{decision} = 0 & y \in (-1-\alpha, \alpha - 1) \bigcup (1-\alpha, 1+\alpha) \end{cases} \tag{2}$$

where $\alpha = \sqrt{\ln(\frac{2\tau^2 e}{\pi})}$.

- 3) When $\tau > \sqrt{\frac{\pi}{2}}$, the decision is

$$\begin{cases} \text{decision} = 1 & y \in (-\infty, -1-\alpha) \bigcup (1+\alpha, +\infty) \\ \text{decision} = 0 & y \in (-1-\alpha, 1+\alpha) \end{cases} \tag{3}$$

where $\alpha = \sqrt{\ln(\frac{2\tau^2 e}{\pi})}$.

Exercise 2.5

a)

$$L(\mathbf{y}) = \prod_{k=1}^{N} \frac{f(y_k|H_1)}{f(y_k|H_0)} = e^{-\lambda \sum_{k=1}^{N}[|y_k - A| - |y_k|]} \tag{4}$$

Then, we can derive the LRT as

$$\frac{1}{N}\sum_{k=1}^{N}[|y_k| - |y_k - A|] \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln \tau}{N\lambda} \triangleq \eta. \tag{5}$$

The lefthand of the equation is the sufficient statistic for the LRT.

- 1) When $y_k \le 0$:

$$|y_k| - |y_k - A| = -A \tag{6}$$

- 2) When $y_k \ge A$:

$$|y_k| - |y_k - A| = A \tag{7}$$

- 3) When $0 \le y_k \le A$:

$$|y_k| - |y_k - A| = 2y_k - A \tag{8}$$

Therefore, we can express the sufficient statistic as follows:

$$S = \frac{1}{N}\sum_{k=1}^{N} c(Y_k - \frac{A}{2}) \tag{9}$$

where $c(\cdot)$ is the symmetric clipping function.

b) When the two hypotheses are equally likely and we seek to minimize the probability of error, so the Bayesian costs are given by $C_{ij} = 1 - \delta_{ij}$, the Bayesian threshold $\tau = 1$, then $\eta = 0$.

c) The clipping function $c(\cdot)$ truncates the high and low values of centered observations $Y_k - \frac{A}{2}$ to prevent them from dominating the sum S.

Exercise 2.7

a)

$$L(k) = \frac{q_1^k(1-q_1)^{n-k}}{q_0^k(1-q_0)^{n-k}} \underset{H_0}{\overset{H_1}{\gtrless}} \tau = \frac{\pi_0}{\pi_1} \tag{10}$$

The LRT is

$$k \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\ln\left(\frac{\pi_0(1-q_0)^n}{\pi_1(1-q_1)^n}\right)}{\ln\left(\frac{q_1(1-q_0)}{q_0(1-q_1)}\right)} \triangleq \eta \tag{11}$$

b) According to the LRT, we derive the four cases of (P_F, P_D) as following table.

η	P_F	P_D
-1	1	1
0	$2q_0(1-q_0)+q_0^2$	$2q_1(1-q_1)+q_1^2$
1	q_0^2	q_1^2
2	0	0

Exercise 2.8

a) The Bayesian test can be expressed as

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} = 2e^{-|y|} \underset{H_0}{\overset{H_1}{\gtrless}} \tau, \tag{12}$$

or equivalently

$$|y| \underset{H_1}{\overset{H_0}{\gtrless}} \ln(\frac{2}{\tau}) \triangleq \eta. \tag{13}$$

b) The probability of false alarm and detection can be expressed as

$$\begin{aligned} P_F &= \int_{-\ln\frac{2}{\tau}}^{\ln\frac{2}{\tau}} \frac{1}{2}e^{-|y|}dy \\ &= \frac{1}{2} \times \left[e^{y}\Big|_{-\ln\frac{2}{\tau}}^{0} - e^{-y}\Big|_{0}^{\ln\frac{2}{\tau}}\right] \\ &= 1 - \frac{\tau}{2} \end{aligned} \tag{14}$$

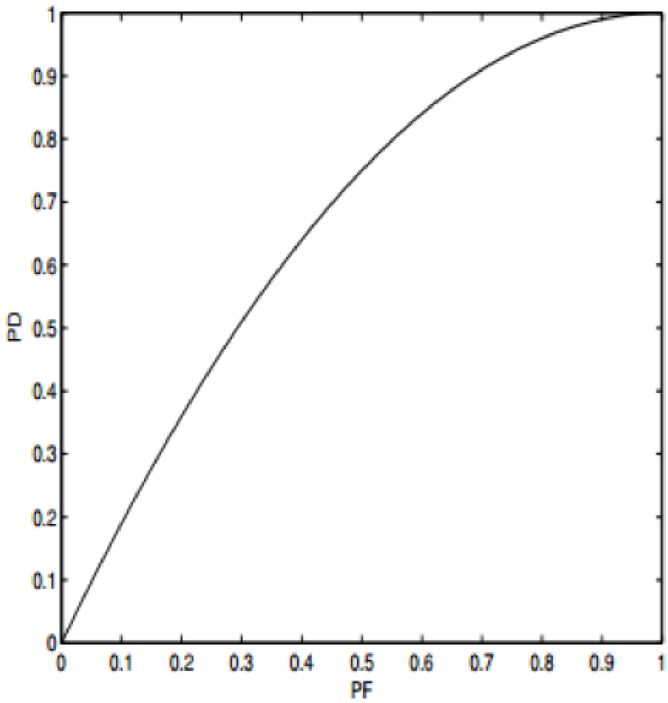
$$\begin{aligned} P_D &= \int_{-\ln\frac{2}{\tau}}^{\ln\frac{2}{\tau}} e^{-2|y|}dy \\ &= \frac{1}{2}e^{2y}\Big|_{-\ln\frac{2}{\tau}}^{0} - \frac{1}{2}e^{-2y}\Big|_{0}^{\ln\frac{2}{\tau}} \\ &= 1 - \frac{\tau^2}{4} \end{aligned} \tag{15}$$

c) Eliminate the τ, we derive

$$P_D = -P_F^2 + 2P_F. \tag{16}$$

Then, plot the ROC as

d) To design a Neyman-Pearson test with probability of false alarm less or equal to α,



we must in fact select $P_F = \alpha$ which, after substitution, yields

$$\tau = 2(1-\alpha). \tag{17}$$

Exercise 2.10

a)

$$L(y) = \begin{cases} 4y & 0 \le y \le \frac{1}{2} \\ 4(1-y) & \frac{1}{2} \le y \le 1 \end{cases} \tag{18}$$

- 1) When $\tau > 2$: y always detect as H_0.
- 2) When $\tau \le 2$:
  - i) When $y \in [0, \frac{1}{2}]$: $y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\tau}{4}$.
  - ii) When $y \in [\frac{1}{2}, 1]$: $y \underset{H_1}{\overset{H_0}{\gtrless}} 1 - \frac{\tau}{4}$.

b)

- 1) When $\tau > 2$: $P_D = P_F = 0$.
- 2) When $\tau \le 2$:

$$P_F = \int_{\frac{\tau}{4}}^{\frac{1}{2}} 1dy + \int_{\frac{1}{2}}^{1-\frac{\tau}{4}} 1dy = 1 - \frac{\tau}{2} \tag{19}$$

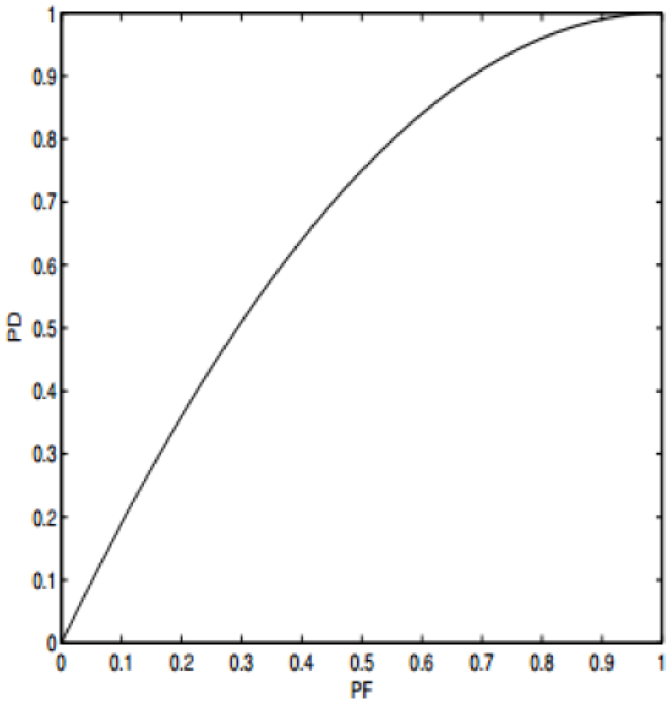
$$P_D = \int_{\frac{\tau}{4}}^{\frac{1}{2}} 4ydy + \int_{\frac{1}{2}}^{1-\frac{\tau}{4}} 4(1-y)dy = 1 - \frac{\tau^2}{4} \tag{20}$$

c) Eliminate the τ, we derive

$$P_D = -P_F^2 + 2P_F. \tag{21}$$

Then, plot the ROC as

d) To design a Neyman-Pearson test with probability of false alarm less or equal to α,



we must in fact select $P_F = \alpha$ which, after substitution, yields

$$\tau = 2(1 - \alpha). \tag{22}$$